

Soliton solution in dilaton - Maxwell gravity

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Abstract

The inverse scattering problem method application to construction of exact solution for Maxwell dilaton gravity system is considered. By use of Belinsky and Zakharov L - A pair the solution of the theory is constructed. The rotating Kerr - like configuration with NUT - parameter is obtained.

1 Model discussed

Recently much attention has been given to the study of gravity models appearing in the low energy limit of string theory [1]–[5]. These models describe various interacting "matter" fields coupled to gravity; one of them considers the system of interacting gravitational, abelian vector and scalar fields with the action

$$^4S = \int d^4x |g|^{\frac{1}{2}} \left[-R + 2(\partial\phi)^2 - e^{-2\phi} F^2 \right], \quad (1.1)$$

where R is the Ricci scalar for the metric $g_{\mu\nu}$, ($\mu = 0, \dots, 3$); $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We will discuss the stationary and axisymmetric model when both the metric and matter fields depend only on two space coordinates.

The four - dimensional line element can be parametrized according to [6]

$$ds_4^2 = g_{\mu\nu} dx^\mu dx^\nu = f(dt - \omega_\varphi d\varphi)^2 - f^{-1} ds_3^2, \quad (1.2)$$

and the three - dimensional one can be taken in the Lewis - Papapetrou form

$$ds_3^2 = h_{mn} dx^m dx^n = e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\varphi^2. \quad (1.3)$$

The system under consideration may be completely described by the Einstein equations defining the metric function γ , as well as by the set of motion equations for the action

$$^2S = \int d\rho dz \rho L, \quad (1.4)$$

where Lagrangian in term of σ - model variables [7] has the form

$$\begin{aligned} L = & \frac{1}{2} f^{-2} [(\nabla f)^2 + (\nabla\chi + v\nabla u - u\nabla v)^2] \\ & - f^{-1} [e^{2\phi} (\nabla u)^2 + e^{-2\phi} (\nabla v)^2] \\ & + 2(\nabla\phi)^2. \end{aligned} \quad (1.5)$$

(∇ is connected with the flat two - metric δ_{ab} .) Here the magnetic u and rotation χ potentials are introduced by "dualization" in three - dimensional formulation of the stationary theory instead of the vector potential \vec{A} and metric function $\vec{\omega}$ respectively:

$$\nabla u = f e^{-2\phi} (\sqrt{2} \nabla \times \vec{A} + \nabla v \times \vec{\omega}) + \kappa \nabla v, \quad (1.6)$$

$$\nabla\chi = u\nabla v - v\nabla u - f^2\nabla \times \vec{\omega}, \quad (1.7)$$

where $v = \sqrt{2}A_0$, and three-dimensional operator ∇ is connected with the metric h_{ij} .

In this letter we will study the case with $\chi = u = 0$, i. e. the pure electric configurations. It can be described by the action [8]

$$^2S = \frac{1}{2} \int d\rho dz \rho \text{Tr}(J^P)^2, \quad (1.8)$$

where $J^P = (\nabla P)P^{-1}$ and the symmetric 2×2 matrix P consists of the "matter" fields:

$$P = \begin{pmatrix} f - v^2 e^{-2\phi} & -ve^{-2\phi} \\ -ve^{-2\phi} & -e^{-2\phi} \end{pmatrix}, \quad (1.9)$$

The set of Euler - Lagrange equations in the form of matrix equation reads

$$\nabla(\rho J^P) = 0, \quad (1.10)$$

and it is for this equation that we will apply the inverse scattering problem method. The corresponding four - metric becomes static; its space part is determined by the function γ

$$\begin{aligned} \gamma_{,z} &= \frac{\rho}{2} \text{Tr}[J_\rho^P J_z^P], \\ \gamma_{,\rho} &= \frac{\rho}{4} \text{Tr}[(J_\rho^P)^2 - (J_z^P)^2]. \end{aligned} \quad (1.11)$$

As it is well known, having one solution of the theory one can obtain another one by applying some kind of symmetry transformation. So by changing the dilaton sign one can construct the pure magnetic field configuration with the same metric. Similarly the complex discrete Bonnor transformation [9] maps the system (1.8) into the one with the Kerr - like space - time metric with NUT parameter.

Thus in this letter we construct the nontrivial asymptotically flat configuration from the trivial fields and space - time metric by using the inverse scattering problem technique, and from this with the help of symmetry transformations we obtain the systems with the other set of field variables.

2 Inverse scattering problem method

As it has been shown by Belinsky and Zakharov, the vacuum stationary axially symmetric gravitational equations may be integrated by use of the inverse scattering problem method as proposed in [10] - [11]. This method allows to obtain the n - soliton configurations from the flat space - time, and in the case of two - soliton solution the Kerr - NUT metric with horizons is constructed .

Let us describe in brief the main aspects of general scheme used in further consideration [11].

The part of vacuum axially symmetric Einstein equations determining the metric of subspace (t, ϕ) reads

$$\nabla(\rho J^g) = 0, \quad J^g = (\nabla g)g^{-1}, \quad (2.1)$$

$(\nabla_i = \partial_i, i = \rho, \phi)$ where g must satisfy the condition

$$\det g = -\rho^2. \quad (2.2)$$

The integration of (2.1) is associated with the L - A pair

$$[\partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2} \partial_\lambda] \psi = \frac{\rho J^g_z - \lambda J^g_\rho}{\lambda^2 + \rho^2} \psi, \quad (2.3)$$

$$[\partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2} \partial_\lambda] \psi = \frac{\rho J^g_\rho + \lambda J^g_z}{\lambda^2 + \rho^2} \psi,$$

where λ is the spectral complex parameter and the function $\psi = \psi(\lambda, \rho, z)$. Then the solution of (2.1) for the symmetric metric matrix g represents as:

$$g(\rho, z) = \psi(0, \rho, z). \quad (2.4)$$

The soliton solutions for the matrix g correspond to the pole divergense in the spectral parameter complex plane for the matrix ψ ; its pole trajectories are determined by

$$\mu_k(\rho, z) = w_k - z \pm [(w_k - z)^2 + \rho^2]^{\frac{1}{2}}, \quad w_k = \text{const} \quad (2.5)$$

for each pole k .

To satisfy the requirement (2.2), it is useful to note, that when g is the solution of Eq. (2.1), the "physical" function $g^{ph} = -\rho(-\det g)^{-\frac{1}{2}}g$ is the solution, too.

Hence, omitting the intermediate calculations, the resulting expression for the metric tensor n - soliton solution has the form:

$$g^{ph}_{ab}(\rho, z) = -\rho^{-n} \left(\prod_{p=1}^n \mu_p \right) [(g_0)_{ab} - \sum_{k,l=1}^n \Gamma_{kl}^{-1} \mu_k^{-1} \mu_l^{-1} N^{(k)}_a N^{(l)}_b], \quad (2.6)$$

where

$$N^{(p)} = g_0 [\psi_0^{-1}(\mu_p, \rho, z)]^T m_0^{(p)}, \quad \Gamma_{kl} = (\rho^2 + \mu_k \mu_l)^{-1} N^{(k)T} g_0^{-1} N^{(l)}, \quad (2.7)$$

and the column $m_0^{(p)}$ consists of arbitrairy constants:

$$m_0^{(p)} = \begin{pmatrix} C_0^{(p)} \\ C_1^{(p)} \end{pmatrix}, \quad (2.8)$$

corresponding to the different characteristics of the source.

3 Exact soliton solution

Now let us apply the inverse scattering problem method to the construction of the static axial - symmetric two - soliton solution for the Maxwell - dilaton gravity model (1.8).

If one takes the matrix Euler - Lagrange motion equation (1.10) and distracts from the physical sence of its components, one can see that its form coinsides with (2.1) for Einstein gravity. The important fact is that the matrix dimation of both this expressions is the same. This gives the reason to believe that the application of the considered above technique allows to obtain an exact solution of the discussed model.

Our system does not have a condition like (2.2), but the asymptotic behavior of the metric and fields raquires

$$P_\infty = \sigma_3. \quad (3.1)$$

To satisfy this we determine the "physical" matrix as $P^{ph} = -(-\det P)^{-\frac{1}{2}}P$, which is also the solution of (1.10). This leads to the restriction $\det P^{ph} = -1$,

in other words for all solutions it must be $f = e^{2\phi}$ (see (1.9)). This limitation being the result of the technique used shows, that the pure nontrivial gravity system or the Einstein - Maxwell system are not contained in the constructed solution.

The initial values of metric and field variables correspond to a flat space - time and a trivial dilaton and electric configurations (3.1). Following [11], we obtain the solution in the Boyer - Lindquist coordinats:

$$\rho = [(r - m)^2 - \sigma^2]^{\frac{1}{2}} \sin \theta, \quad z - z_1 = (r - m) \cos \theta, \quad (3.2)$$

where the new constants $\sigma = \frac{1}{2}(w_1 - w_2)$ and $z_1 = \frac{1}{2}(w_1 + w_2)$ are introduced (see (2.5)); as one can see below, m corresponds to the mass of the source. In analogy with the pure gravity case we impose the conditions on the arbitrary constants (2.8):

$$C_1^{(1)}C_0^{(2)} - C_0^{(1)}C_1^{(2)} = \sigma, \quad C_1^{(1)}C_0^{(2)} + C_0^{(1)}C_1^{(2)} = -m, \quad (3.3)$$

$$C_1^{(1)}C_1^{(2)} - C_0^{(1)}C_0^{(2)} = a, \quad C_1^{(1)}C_1^{(2)} + C_0^{(1)}C_0^{(2)} = -b, \quad (3.4)$$

where $\sigma^2 = m^2 - b^2 + a^2$.

If we introduce the notation

$$\Delta = r^2 - 2mr + b^2 - a^2, \quad \delta^2 = r^2 - (b - a \cos \theta)^2, \quad (3.5)$$

then the components of matrix P can be presented in the form

$$f = e^{2\phi} = \frac{\Delta + a^2 \sin^2 \theta}{\delta^2}, \quad v = 2 \frac{-br + m(b - a \cos \theta)}{\delta^2}. \quad (3.6)$$

The four - dimensional line element corresponds to diagonal metric with space part resembling the Kerr one:

$$\begin{aligned} ds_4^2 &= \frac{\Delta + a^2 \sin^2 \theta}{\delta^2} dt^2 - \frac{\delta^2}{\Delta + a^2 \sin^2 \theta} \Delta \sin^2 \theta d\varphi^2 \\ &- \frac{\Delta + a^2 \sin^2 \theta}{\Delta + \sigma^2 \sin^2 \theta} \left[\frac{\delta^2}{\Delta} dr^2 + \delta^2 d\theta^2 \right]. \end{aligned} \quad (3.7)$$

Hence one can see that m is the mass of the source and b is proportional to the electric charge.

As it was noted before, it is interesting to construct one field configuration from the other using some symmetry transformation. By changing the dilaton sign from the electric solution one can obtain the pure magnetic solution with the same metric and $f = e^{-2\phi}$, $u = v$, where f and v are determined by (3.6).

Another interesting case is connected with the complex discrete transformation allowing to obtain the rotating system with NUT - parameter. The Bonnor transformation

$$v \rightarrow -i\chi \quad (3.8)$$

leads to a complex changing of parameters a and b corresponding now to a rotating moment and a NUT - parameter of the source. Given transformation maps the effective Lagrangian $L(f, v, \phi)$ into the one $L(f, \chi) = 2L_E$, where L_E relates to the vacuum gravity. One can see that this Lagrangian appears in the Einstein - Maxwell - dilaton theory with pseudoscalar Peccei - Quinn axion field (EMDA) [12]. (As it is known, the theory (1.1) is not always a subsystem of EMDA). Thus one can leave the framework of the theory under consideration (1.1) and obtain the EMDA particular solution that presents six "moduli" field expressed in terms of two variables f and χ . One of this possible models corresponding to the special solution anzats is considered in [13]; as an example we would like to present the symplest case with $f = -e^{-2\phi}$ and axion field $\kappa = \chi$, related to the system without vector fields.

Then the field functions have the form:

$$f = -e^{-2\phi} = \frac{\Delta - a^2 \sin^2 \theta}{\delta^2}, \quad \chi = \kappa = 2 \frac{-br + m(b - a \cos \theta)}{\delta^2}. \quad (3.9)$$

Using (1.7) we obtain

$$\omega_\varphi = \frac{2}{\Delta - a^2 \sin^2 \theta} [b \cos \theta \Delta - a \sin^2 \theta (mr + b^2)], \quad (3.10)$$

which defines the metric component $g_{t\varphi}$. Thus the resulting four - metric becomes:

$$\begin{aligned} ds_4^2 = & \frac{\Delta - a^2 \sin^2 \theta}{\delta^2} [dt - \omega_\varphi d\varphi]^2 - \frac{\delta^2}{\Delta - a^2 \sin^2 \theta} \Delta \sin^2 \theta d\varphi^2 \\ & - \frac{\Delta - a^2 \sin^2 \theta}{\Delta + \sigma^2 \sin^2 \theta} \left[\frac{\delta^2}{\Delta} dr^2 + \delta^2 d\theta^2 \right]. \end{aligned} \quad (3.11)$$

It is easy to see that this expression is close to the Kerr - NUT metric. The difference is associated with the duplication of L_E ; this leads to the fact that the metric function $\gamma = 2\gamma_E$ (see (1.11)), changing the three - dimensional metric (1.3). As the result one has the dependence of the metric on the coordinate θ in the spherically symmetric case (when $a = b = 0$), this also concerns the metric (3.7).

At the last time one can note that the equation (1.10) is invariant under the transformation

$$P \rightarrow P^{-1}. \quad (3.12)$$

It is easy to prove that the solution arising after applying of (3.12) coincides with the one constructed above with the additional replacement $z \rightarrow -z$ and $b \rightarrow -b$.

4 Discussion

In this letter by use of the inverse scattering problem method the exact stationary axially symmetric solution of interacting electric - dilaton gravity system is constructed. This becomes possible because of the suitable matrix dimension description of the theory under consideration and as a result of the motion equations. Actually this fact plays an important role: so, the direct application of above mentioned technique for the case of the Einstein - Maxwell - dilaton - axion gravity describing by the four - dimensional matrix does not allow to obtain the solution with "good" asymptotics for metric and fields. The technique used fixes the ansatz not containing the nontrivial gravity or Maxwell - gravity systems as subsystems.

This demonstrates the possibility of soliton solution construction for gravity system with fields described by two - dimensional matrices, and we would hope in further to generalize this for the case of arbitrary degree of freedom number.

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